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Extended Surface Heat Rejection Accounting for Stochastic Sink Temperatures

T. Yoshimura*
University of Tokushima, Tokushima, Japan
and
A. Campo†
Universidad Simón Bolívar, Caracas, Venezuela

The principal objective of this investigation was to determine the magnitude of the local temperature variance along an extended surface arising from the influence of stochastic ambient temperatures. The supporting data for the ambient temperature are given by two statistical quantities, the mean and the variance. An analysis of the problem is performed, the special feature of which is the accounting of the ambient temperature variance. This parameter serves as the input for calculation of the resulting variance of the local temperature along the extended surface. Results are given as plots of the temperature band vs position in terms of the participating thermal parameters. From numerical calculations, the values of both maximum and minimum heat rejection to the ambient are presented concurrently. Under certain limiting conditions the present solutions are found to reduce to the classic results based upon deterministic analysis.

Nomenclature

b = half-width of extended surface
E = expectation
$\bar{h}$ = mean convective coefficient
k = thermal conductivity
L = length of extended surface
N_c = convective parameter, $\bar{h}L^2/kb$
q_{mn} = function defined in Eq. (14)
Q = heat-transfer rate
t = time
T = temperature
T_a = ambient temperature
T_b = base temperature
v = covariance of temperature in Eq. (6)
V = transformed covariance defined in Eq. (13)
W = variance of the ambient temperature
x = coordinate
Z = depth of extended surface
α = thermal diffusivity
δ = Dirac's delta function
η = dimensionless coordinate, x/L
ξ = coordinate for the covariance in Eq. (6)
σ = time defined in Eq. (3)
θ = dimensionless temperature, T/T_b
$\hat{\theta}$ = dimensionless mean temperature
θ_a = dimensionless ambient temperature, T_a/T_b
$\hat{\theta}_a$ = mean of θ_a
$\tilde{\theta}$ = temperature deviation, $\theta_a - \hat{\theta}_a$
τ = dimensionless time, $\alpha t/L^2$
Ω = dimensionless heat transfer rate, $Q/(kbZT_b/L)$

Subscripts

det = deterministic
max = maximum
min = minimum

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*Associate Professor Dept. of Mechanical Engineering.
†Professor, Depto. de Termodinámica.

Introduction

THE factors controlling the thermal performance of extended surfaces are well established and have been presented in a vast number of textbooks and journal publications in the area of thermal engineering. In addition to the usual heat-transfer problems associated with extended surfaces, there are other important problems encountered in such devices that deserve some analysis. In particular, one of these problems deals with the behavior of the ambient temperature surrounding the extended surface which is considered in detail in this paper. This external temperature is customarily treated as a quantity known a priori in both steady- and unsteady-state studies.

There are situations where extended surfaces may be subjected to a wide variety of unsteady conditions due to temporal changes of the ambient temperature. Hence, two possibilities arise for the ambient temperature being: 1) deterministic, a known function of time, or 2) stochastic, an unknown function of time. Specifically, these two categories may be explained in terms of the temperature-time description of the ambient fluid which is envisioned as consisting of two major components: the mean value and its corresponding fluctuations. The dominance of the mean value over the fluctuations depend to a great extent on the relative magnitude of the temperature deviations around the mean. Therefore, when these deviations are considered to be relatively small, their influence can be discarded such that the ambient temperature is assumed to be identical to its mean with variance equal to zero. This is the normal procedure in design calculations of heat-exchange equipment where the appropriate predictive analysis of the temperature profile and heat-transfer rates rely exclusively on the mean value of the ambient temperature. Of course, this mean value can be represented by a constant quantity or by a functional relation of time depending on the physics of the problem. Such a procedure is traditionally called deterministic.

The present work is concerned with the calculation of the thermal behavior and the heat exchange from an extended surface in the presence of an ambient fluid, the temperature of which is an unknown function of time. This corresponds to a situation where fluctuations of the ambient temperature are not small when compared to its mean value. Under these circumstances, the mathematical representation for this class of problems requires that the ambient temperature be specified by two statistical quantities simultaneously, i.e., the mean and the variance. This behavior is interpreted as fully stochastic.

Examination of the heat-transfer literature reveals that previous studies on extended surfaces have been carried out under the simplifying assumption of negligible fluctuations in ambient temperature. The reason for this is largely because of the intrinsic complexity involved in dealing with these fluctuations. Apparently, no existing work provides a comprehensive assessment of this phenomenon. Such information has both qualitative and quantitative importance and, of course, is essential for justifying the use of zero variance for the ambient temperature in investigations related to extended surface heat transfer.

The present treatment approaches the general problem in a different way wherein both components of the ambient temperature defined by the mean and its variance are considered equally important for the complete description of the cooling process. These data are utilized for the calculation of the temperature profile along the extended surface with the aid of statistical analysis. Hence, the resulting information has to be presented also in terms of the calculated mean and variance of the temperature profile. The results of this investigation may be applicable to certain engineering problems, such as temperature measurements in pipe flows where the fluid temperature changes arbitrarily with time, and also to heat-transfer equipment exposed to atmospheric conditions that are abruptly altered by the weather.[1,2]

The successful application of this study depends heavily upon knowledge of the statistical parameters of the ambient temperature. The computed results for the extended surface should be of special usefulness when some measurements of the ambient temperature are not available or are incomplete.

Aside from the references already cited, there appears to be very little published on stochastic heat transfer. The only publication that is somewhat related to the present study is Hung's paper examining stochastic base temperature variations in extended surfaces.[3]

Governing Equations

The physical situation analyzed here deals with an extended surface of a rectangular profile attached to a wall. The thermal interaction between the exposed surface and the ambient temperature takes place by convection and it is assumed that the convective coefficient is uniform and constant on the surface. This condition was chosen with the idea of minimizing the number of parameters while maintaining the essential features of the problem. The input to this physical system, i.e., the ambient temperature $\theta_a(\tau)$, is considered stochastic because its variation cannot be defined completely by a prescribable function. Two quantities are necessary for the postulation of the problem within the framework of stochastic theory: the mean and the variance of the ambient temperature. Correspondingly, the temperature distribution on the extended surface remains unspecified, presenting some uncertainties which have to be expressed in terms of a mean temperature distribution and a variance of the temperature distribution. These results can be interpreted as an envelope wherein the variance distribution gives an allowance for the upper and lower bounds of the accuracy of the mean temperature distribution.

The following analysis presents a systematic method for selecting the necessary bounds for the thermal performance of an extended surface accounting for the effects of stochastic ambient temperature. The local temperature along the extended surface is described by the dimensionless energy equation

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial^2 \theta}{\partial \eta^2} - N_c[\theta - \theta_a(\tau)] \tag{1}$$

In this equation, $\theta_a(\tau)$ is assumed to be a stationary normal process having mean and covariance (variance is a particular case) defined by

$$E[\theta_a(\tau)] = \hat{\theta}_a \tag{2}$$

and

$$E[\tilde{\theta}_a(\sigma)\tilde{\theta}_a(\tau)] = W\delta(\sigma - \tau) \tag{3}$$

respectively.

The linearity of the physical system given by Eqs. (1-3) implies that the mean and covariance of the temperature distribution are the only statistical parameters needed for the complete solution of the problem. Alternatively, the mean and covariance are normally called the first and second moments of $\theta(\eta,\tau)$.

Since the primary emphasis of this study is not focused on the transient initial stage, the standard spatial boundary conditions imposed on Eq. (1) are

$$\theta(0,\tau) = 1 \qquad \frac{\partial \theta}{\partial \eta}(1,\tau) = 0 \tag{4}$$

together with the initial condition $\theta(\eta,0) = 1$.

At this point, the systematic calculation procedure is divided in two main parts. First, the mean temperature distribution $\hat{\theta}(\eta,\tau)$ obeys Eq. (1) subject to the boundary conditions of Eq. (4) when θ is replaced by $\hat{\theta}$ and $\theta_a(\tau)$ is replaced by $\hat{\theta}_a$, respectively, in both equations. Second, the descriptive equation for the covariance $v(\xi,\eta,\tau)$ at any two points is derived in the Appendix and expressed as follows

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial \xi^2} + \frac{\partial^2 v}{\partial \eta^2} - 2N_c v + N_c^2 W \tag{5}$$

where

$$v(\xi,\eta,\tau) = E[\tilde{\theta}(\xi,\tau)\tilde{\theta}(\eta,\tau)] \tag{6}$$

and the applicable boundary and initial conditions are given by

$$v(\xi,0,\tau) = v(0,\eta,\tau) \tag{7}$$

$$\frac{\partial v}{\partial \eta}(\xi,1,\tau) = \frac{\partial v}{\partial \xi}(1,\eta,\tau) = 0 \tag{8}$$

$$v(\xi,\eta,0) = 0 \tag{9}$$

Solutions of the just-outlined problem depend on the values of three parameters, the mean $\hat{\theta}_a$ and variance W of the ambient temperature and the convective parameter N_c.

The solution for the mean temperature distribution $\hat{\theta}(\eta,\tau)$, determined via the superposition principle, is expressed as follows[4]

$$\hat{\theta}(\eta,\tau) = \hat{\theta}_I(\eta) + \hat{\theta}_{II}(\eta,\tau) \tag{10}$$

where $\hat{\theta}_I$ denotes the steady solution

$$\hat{\theta}_I = \hat{\theta}_a + (1 - \hat{\theta}_a)\frac{\cosh\sqrt{N_c}(1-\eta)}{\cosh\sqrt{N_c}} \tag{11}$$

and $\hat{\theta}_{II}$ denotes the unsteady solution

$$\hat{\theta}_{II} = \sqrt{2}(1 - \hat{\theta}_a)\sum_{n=1}^{\infty} \frac{N_c \phi_n(\eta)}{\alpha_n(\alpha_n^2 + N_c)} e^{-(\alpha_n^2 + N_c)\tau} \tag{12}$$

respectively. In Eq. (12) the eigenfunctions are calculated from the relation $\phi(\eta) = \sqrt{2}\sin\alpha_n\eta$ while the eigenvalues α_n are equal to $\alpha_n = (2n+1)\pi/2$; $n = 1,2....$[4]

To compute the resulting covariance from the set of Eqs. (5-9), it is helpful to introduce a change of variable

$$V(\xi,\eta,\tau) = e^{2N_c\tau} v(\xi,\eta,\tau) \quad (13)$$

The solution of the transformed problem in terms of this new variable V is assumed to be of the following form

$$V(\xi,\eta,\tau) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} q_{mn}(\tau)\phi_n(\xi)\phi_m(\eta) \quad (14)$$

Inserting Eq. (14) into the appropriate differential equation and boundary conditions and after some manipulations, it is obtained that

$$q_{mn}(\tau) = \frac{2N_c^2 We^{-(\alpha_n+\alpha_m)\tau}}{\alpha_n\alpha_m(\alpha_n^2+\alpha_m^2+2N_c)}[e^{(\alpha_n^2+\alpha_m^2+2N_c)\tau} - 1] \quad (15)$$

where $q_{mn}(0)=0$. With this and using Eq. (13), the covariance distribution $v(\xi,\eta,\tau)$ may be written in a condensed form as follows:

$$v(\xi,\eta,\tau) = 4N_c^2 W \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(\sin\alpha_n\xi)(\sin\alpha_m\eta)}{\alpha_n\alpha_m(\alpha_n^2+\alpha_m^2+2N_c)} \times [1 - e^{-(\alpha_n^2+\alpha_m^2+2N_c)\tau}] \quad (16)$$

At this point, it should be emphasized that conceptually speaking the variance distribution is computed from Eq. (16) when $\xi=\eta$ only. The physical significance of this quantity $v(\eta,\eta,\tau)$ stems from the fact that it provides an allowance for the computed mean temperature distribution. On the other hand, Eq. (16) in its present form, i.e., for $\xi \neq \eta$, gives a covariance distribution which has no physical meaning for the calculation of the mean temperature distribution. Therefore, numerical computations of the covariance are omitted altogether. In light of Eq. (16), the quantity v represents an algebraic expression which contains two of the parameters of the problem, N_c and W. With these given parameters, all the inputs to the right-hand side of Eq. (16) are available. Additionally, it is observed that there exists a linear relationship between W and v due to the linearity of Eq. (1). This relation may be explained by stating that the variance of the ambient temperature W has to be considered the most important factor in determining the confidence interval of the mean temperature distribution within the framework of stochastic problems.

Results and Discussion

Numerical results for the mean and variances associated with local temperatures along an extended surface accounting for a stochastic ambient temperature were obtained for a broad range of conditions. As mentioned earlier, the mathematical model allows for the calculation of both the initial unsteady and steady regimes, but only the latter will be reported because of its significance in practical applications to engineering problems. Accordingly, variations were assigned to the convective parameter N_c and the variance of the ambient temperature W, while the mean of the ambient temperature $\hat{\theta}_a$ was kept fixed at an intermediate value of 0.5 for all cases tested. The results for the variance v and the envelope for the temperature θ are presented in Figs. 1 and 2, respectively. In each figure the dimensionless coordinate η is the abscissa variable, and N_c and W are parameters of the individual graphs. Two extreme cases were selected for illustration purposes: one dealing with a low N_c and high W, i.e., $N_c=1$ and $W=0.1$; and the other related to a high N_c and low W, i.e., $N_c=5$ and $W=0.025$.

Figure 1 shows a typical pattern of local variances along the extended surface. Here the solid line represents the first case of $N_c=1$ and $W=0.1$, while the broken line illustrates the

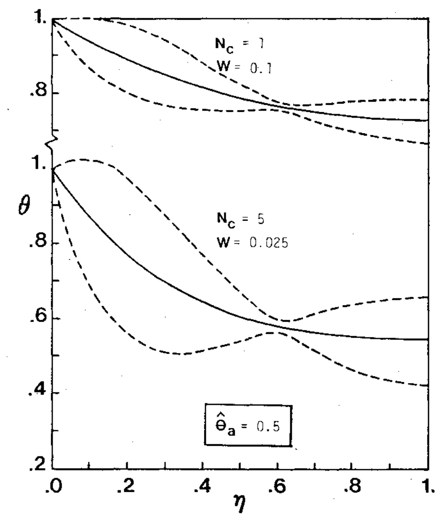

Fig. 1 Variance distributions of local temperatures.

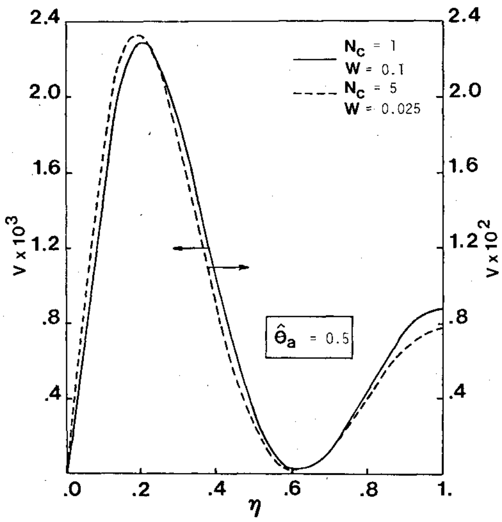

Fig. 2 Maximum and minimum temperature profile based upon 95% probability.

situation for $N_c=5$ and $W=0.025$. From an examination of Fig. 1 certain general trends are in evidence. The shapes of both curves are almost identical, although the variances for the second case are one order of magnitude larger than the counterpart for the first case. The main feature projected by this figure is the substantial amount of temperature fluctuation that can occur along the first half of the extended surface. Hence, it is completely appropriate to discuss both cases concurrently. From the theoretical point of view, it should be mentioned that the variance distribution $v(\eta,\eta)$ is governed by Eq. (16) consisting of a constant source term $N_c^2 W$ and also a variance-dependent sink term $2Nc_v$. Accordingly, the portion close to the base of the extended surface seems to be totally dominated by the effects of the source term. On the other hand, the remaining regions of the body appear to be influenced by the presence of the sink term because the variance magnitudes are not as high as before. The extent of the region where this pattern occurs remains almost independent of the magnitude of the parameters N_c and W.

For practical applications these findings may be interpreted in a better way when they are linked to the mean temperature distribution giving the permissible margin of variations for local temperatures along the extended surface. These results are summarized in Fig. 2. Moreover, the linearity of Eq. (1) indicates that the mean temperature $\hat{\theta}$ depends upon $\hat{\theta}_a$ only and the steady-state component of the mean temperature distribution $\hat{\theta}_I$ represented by Eq. (11) is obtained directly from standard textbooks on heat transfer. For illustration purposes, the resulting mean temperature profile $\hat{\theta}(\eta)$ expressed by Eq. (10) is drawn by solid lines and the permissible upper and lower bounds are drawn by broken lines. In this context, the confidence limits for a 95% probability are computed from the formula[5]

$$\theta = \hat{\theta} \pm 1.96\sqrt{v} \tag{17}$$

This equation gives rise to a representation in terms of an envelope limiting the maximum and minimum temperatures on the extended surface. Inspecting the upper envelope of Fig. 2 ($N_c = 1$), it is observed that it is relatively narrow and, of course, will be even narrower for values of $W<0.1$. This means that these cases can be analyzed by deterministic techniques without incurring significant errors. On the contrary, the lower envelope ($N_c = 5$) shows a wider temperature variation even for small W ($=0.025$). Hence, the allowance for these temperature variations are higher and the envelopes tend to increase as $W>0.025$. Therefore, under these circumstances the stochastic approach is the only reliable tool for the solution of these problems because of the uncertainties associated to the mean temperature distribution.

It is observed in Fig. 2 that the temperature limits do not decrease monotonically with position η. The appearance of a shrinkage in the neighborhood of $\eta = 0.6$ is caused by the behavior of the variance already discussed with Fig. 1. The probable physical reason for this phenomenon is the fact that a fluctuation near the middle of the extended surface will tend to be diminished by the heat flow to or from the right and left directions. Conversely, a fluctuation near the end will not tend to be averaged as much because of the fixed adiabatic boundary condition. Hence, the heat is allowed to flow in only one direction.

In order to perform its function effectively, an extended surface has to remove a certain amount of heat from the body to which it is attached. Therefore, calculation of the temperature distribution is just an intermediate and inevitable step in the computation of a crucial quantity giving the heat-transfer rate. Although the thermal efficiency is traditionally the main focus in the presentation of these results, there are good reasons for not doing so. The essential fact is that in the present instance, the heat-transfer rate contains three unknowns (N_c, $\hat{\theta}_a$, and W) and consequently the standard thermal efficiency depends upon N_c only. On the other hand, in the conventional analysis, the rate of heat transfer under steady-state conditions is given by a single number. This procedure may provide erroneous results if utilized in situations where the fluctuations of the ambient temperature cannot be omitted, because it tends to underestimate or overestimate the rejected heat transfer. Within the framework of stochastic considerations, it has been demonstrated that temperature depressions or elevations occur along the extended surface. Therefore, integration of the upper and lower curves of the temperature band in Eq. (17) provides the maximum and minimum heat transfer that the extended surface is really allowed to carry out. Hence, the expression for the heat-transfer rate is given by

$$Q = \bar{h}P\int_0^L (T - \hat{T}_a)\,dx \tag{18}$$

Table 1 Heat transfer results for $\hat{\theta}_a = 0.5$

Case	Ω_{max}	Ω_{det}	Ω_{min}
$N_c = 1$, $W = 0.1$	0.814	0.760	0.708
$N_c = 5$, $W = 0.025$	2.800	2.200	1.570

The dimensionless form of this equation becomes

$$\Omega = 2N_c\int_0^1 (\theta - \hat{\theta}_a)\,d\eta \tag{19}$$

where $\Omega = Q/(kbZT_b/L)$ and the integrand is represented as follows

$$(\theta - \hat{\theta}_a) = (1 - \hat{\theta}_a)\frac{\cosh\sqrt{N_c}(1-\eta)}{\cosh\sqrt{N_c}}$$

$$\pm 1.96\left[4N_c^2 W \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{(\sin\alpha_n\eta)(\sin\alpha_m\eta)}{\alpha_n\alpha_m(\alpha_n^2+\alpha_m^2+2N_c)}\right] \tag{20}$$

The numerical computation of Eq. (19) is performed using Simpson's rule and the final results are condensed in Table 1. From an overall examination of this table, the combination of the convective parameter and the variance of the ambient temperature is seen to have a dramatic influence in the heat transfer interval $\Delta\Omega = \Omega_{max} - \Omega_{min}$. The impact of N_c and W are readily noticeable by comparing the temperature curve for $W>0$ with that for $W=0$. Such a comparison shows that the extent of heat-transfer deviations grows rapidly with increasing $N_c^2 W$. That is, for the first case $N_c^2 W = 0.1$ and $\Omega = 0.106$ units, while for the second case $N_c^2 W = 0.63$ and $\Omega = 1.23$ units. Accordingly, to put this finding in perspective, it may be noted that Ω responds to changes in both N_c and W but, as seen from Eq. (20), $\theta - \hat{\theta}_a$ varies with N_c linearly while the variation with W is of the form $\sqrt{W}$.

Appendix: The Covariance Equation

The partial derivative of the covariance $v(\xi,\eta,\tau)$ with respect to time is expressed by the relation

$$\frac{\partial v}{\partial \tau}(\xi,\eta,\tau) = \frac{\partial}{\partial \tau}E[\tilde{\theta}(\xi,\eta)\tilde{\theta}(\eta,\tau)] = E\left[\tilde{\theta}(\eta,\tau)\frac{\partial\tilde{\theta}}{\partial\tau}(\xi,\eta)\right]$$

$$+E\left[\tilde{\theta}(\xi,\tau)\frac{\partial\tilde{\theta}}{\partial\tau}(\eta,\tau)\right] \tag{A1}$$

Since by definition $\tilde{\theta} = \theta - \hat{\theta}$, the combination of Eqs. (1) and (6) yields

$$\frac{\partial\tilde{\theta}}{\partial\tau}(\eta,\tau) = \frac{\partial^2\tilde{\theta}}{\partial\eta^2} - N_c[\tilde{\theta}(\eta,\tau) - \tilde{\theta}_a(\tau)] \tag{A2}$$

Substituting Eq. (A2) into the second term on the right-hand side of Eq. (A1) results in

$$E\left\{\tilde{\theta}(\xi,\tau)\left[\frac{\partial^2\tilde{\theta}}{\partial\eta^2}(\eta,\tau) - N_c\tilde{\theta}(\eta,\tau) + N_c\tilde{\theta}_a(\tau)\right]\right\}$$

$$= \frac{\partial^2 v}{\partial\eta^2}(\xi,\eta,\tau) - N_c v(\xi,\eta,\tau) + N_c E[\tilde{\theta}(\xi,\tau)\tilde{\theta}_a(\tau)] \tag{A3}$$

According to Ref. 6 the last term on the right-hand side of Eq. (A3) becomes

$$N_c E[\tilde{\theta}(\xi,\tau)\tilde{\theta}_a(\tau)] = \frac{N_c W}{2} \tag{A4}$$

Therefore, the second term on the right-hand side of Eq. (A1) is finally written as

$$\frac{\partial^2 v}{\partial\eta^2}(\xi,\eta,\tau) - N_c v(\xi,\eta,\tau) + \frac{N_c W}{2} \tag{A5}$$

Repeating the same operations for the first term on the right-hand side of Eq. (A1) and collecting all equivalent terms, the resulting equation corresponds to Eq. (5).

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